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Non-perturbative gauge groups and local mirror symmetry

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Abstract: We analyze D-brane states and their central charges on the resolution of $\mathbb{C}^2/\mathbb{Z}_n$ by using local mirror symmetry. There is a point in the moduli space where all $n(n-1)/2$ branches of the principal component of the discriminant locus coincide. We argue that this is the point where compactifications of type-IIA theory on a K3 manifold containing such a local geometry acquire a non-perturbative gauge symmetry of the type A_{n-1}. This analysis, which involves an explicit solution of the GKZ system of the local geometry, explains how the quantum geometry exhibits all positive roots of A_{n-1} and not just the simple roots that manifest themselves as the exceptional curves of the classical geometry. We also make some remarks related to McKay correspondence.

Keywords: D-branes, String Duality, Discrete and Finite Symmetries.

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1. Introduction

One of the cornerstones of the second string revolution is the conjecture that there exist a non-perturbative duality between heterotic string theory compactified on a four-torus and type-IIA theory on a K3 surface [1]. This conjecture is based on the equivalence of the moduli spaces, the existence of soliton solutions in type IIA that are believed to be heterotic strings [2, 3], relations to type IIB on coinciding 5-branes [4, 5] and arguments involving the algebras of BPS states in both theories [6]. The most striking prediction of this duality is that type-IIA theory should acquire non-perturbative gauge symmetries at special points in the moduli space. It was soon realized that the occurrence of gauge symmetry enhancement should be connected with ADE type singularities in the compactification geometry [7]. As pointed out already in [7] there are problems with the assumption that gauge symmetry enhancement might occur at the orbifold point in the moduli space where the classical geometry becomes singular, because at the orbifold point the conformal field theory remains well behaved. Indeed it was shown in [8] that, at least in some cases, the orbifold point and the point of gauge symmetry enhancement do not coincide because of the discrepancy between classical geometry and quantum geometry which leads to a non-vanishing B-field at the orbifold point.

A non-perturbative gauge symmetry of ADE type is assumed to arise in the following manner. The Cartan subgroup arises from the usual Kaluza-Klein modes, and the roots of the Lie algebra come from D-brane states that become massless at the point of gauge symmetry enhancement. Apart from the aforementioned problem with the orbifold point the picture of branes wrapping cycles of the classical geometry suffers from the fact that the classical geometry of the resolution of the singularity contains only the cycles corresponding to the simple roots. This raises the question of why certain linear combinations of large volume D-brane states (those whose coefficients are the ones that express positive roots in terms of simple roots) should be considered to be independent states and others not.

This is the problem that we want to address in this paper. We examine the moduli space of type IIA compactified on a resolution of $\mathbb{C}^2/\mathbb{Z}_n$ in detail, using local mirror symmetry and results on central charges of D-branes. In particular we give an explicit solution of the GKZ system corresponding to this geometry. This analysis leads to exact expressions for central charges and hence quantum volumes of cycles. We find that there is a point in the quantum Kähler moduli space of the resolution where all $(n^2-n)/2$ branches of the principal component of the discriminant locus meet. These branches and, moreover, certain D-brane states becoming massless along them, are in one to one correspondence with the complete set of positive roots of A_{n-1}. As expected [4] the monodromy group of these states when taken around the various branches in the vicinity of the gauge enhancement point is the Weyl group of A_{n-1}. Along the way we will also discuss certain issues related to McKay correspondence. Roughly our results can be stated in the following way. Classical holomorphic cycles (which are relevant in the large radius limit) correspond to the simple roots of A_{n-1} whereas the quantum geometry features all positive roots. While the orbifold point is connected with the representation theory of $\mathbb{Z}_n$ the point of enhanced gauge symmetry knows about A_{n-1} and its Weyl group S_n.

The structure of this paper is as follows. In the next section we outline the general strategy for analyzing D-brane central charges of toric Calabi-Yau spaces with the help of local mirror symmetry and summarize certain expectations on these models, in particular if they are of orbifold type. In section 3 we perform this analysis for the case of the resolution of $\mathbb{C}^2/\mathbb{Z}_n$. We find that in order to solve the GKZ system it is useful to change to coordinates different from the ones that are ususally used (the transformation to these coordinates is given in an appendix). We find the precise form of the matching between the classical D-brane states at large volume and the quantum corrected central charges as determined by the GKZ solutions. After some remarks concerning conjectures related to McKay correspondence we analyze the behaviour of the D-brane states in the region where the various branches of the principal component of the discriminant locus come together. Finally a brief summary and discussion of these results is given.

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2. The quantum moduli space of a toric Calabi-Yau manifold

In order to analyze the quantum moduli space and hence the spectrum of D-brane states and central charges of a toric Calabi-Yau manifold X it is useful to follow the steps outlined here. This is a summary of various useful results and conjectures on local mirror symmetry, D-brane central charges, monodromy and McKay correspondence; for a more detailed presentation in a similar context see [9].

1. Analyze the geometry of X; in particular, find the Mori cone (the cone of effective curve classes) and its dual, the Kähler cone. There exists an efficient toric method for doing this. Determine the toric data of the local mirror $\tilde{X}$ from those of X. This amounts to using the monomial-divisor mirror map to assign a monomial M_i to every toric divisor D_i of X. Then $\tilde{X}$ is the hypersurface determined by a general polynomial $\sum a_i M_i$ that is a sum of these monomials. The linear equivalence relations among the toric divisors of X, as encoded in the secondary fan of X, determine multiplicative relations among the coefficients a_i, ensuring that the complex structure moduli space of $\tilde{X}$ again allows a toric description in terms of the variety determined by the secondary fan.

2. The quantum moduli space of complexified Kähler classes on X is identified with the complex structure moduli space of $\tilde{X}$. As shown in [10] the quantum volumes of compact cycles in X have to obey the GKZ differential system corresponding to $\tilde{X}$. More generally the central charges of D-branes on X wrapping even cycles must have this property.

3. In order to match the GKZ solutions with the data of the classical geometry of X, consider the large volume limit on X which corresponds to the large complex structure limit on $\tilde{X}$. There is a simple procedure for determining the large complex structure coordinates z_i that vanish in this limit in terms of the homogeneous moduli space coordinates a_i by using toric data associated with the Mori cone of X. Depending on the level of sophistication, a D-brane may be described as a holomorphic cycle (possibly with a vector bundle on it), a K-theory class, or an object in the derived category of coherent sheaves. Independently of the chosen description it is always possible to assign a Chern character to a D-brane, which is all we need in the following. In the large volume limit the central charge of a D-brane S is determined by [11, 12]

$$Z^{\mathrm{lv}}(t_i; S) = -\int_X e^{-\sum t_i T_i} \mathrm{ch}(S) \sqrt{\mathrm{Td}(X)}\,, \tag{2.1}$$

where $\sum t_i T_i$ is the expansion of the complexified Kähler class $B + iJ$ in terms of the generators T_i of the Kähler cone. In particular we may take S to be the push-forward (to X) of the structure sheaf of some cycle in X. Choosing such

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cycles to be the curves C_i dual to the complexified Kähler classes T_i leads to

$$t_i - 1 = \frac{\ln z_i}{2\pi i} + O(z)\,. \tag{2.2}$$

Given a basis of GKZ solutions in terms of the z_i (which behave like polynomials in the $\ln z_i$ near $z = 0$) this affords a matching between geometrical objects (or D-branes) and the corresponding central charges.

4. If the GKZ solutions are not holomorphic along some codimension one locus in the moduli space, they may undergo monodromy. Along the large complex structure divisors $z_i = 0$ classical reasoning applies and we have $t_i \to t_i + 1$, corresponding to tensoring S with $\mathcal{O}(T_i)$. More importantly, there is a codimension one locus determined by a polynomial equation in the a_i called the principal component of the discriminant locus along which $\tilde{X}$ develops singularities. There is a natural pairing

$$\langle R, S\rangle = \int_X \mathrm{ch}(R^*)\,\mathrm{ch}(S)\,\mathrm{Td}(X)\,, \tag{2.3}$$

between K-theory classes R and S on a Calabi-Yau manifold X; if X is non-compact at least one of these classes should be compactly supported. It is believed that monodromy around some branch of the principal component is determined by a Fourier-Mukai type transformation

$$\mathcal{F} \to \mathcal{F} - \langle S, \mathcal{F}\rangle S\,, \tag{2.4}$$

where S is a suitable object that becomes massless along this branch. In particular, for compact X it is believed that $S = \mathcal{O}_X$, the structure sheaf of X.

If X is a resolution of a space of the form $\mathbb{C}^d/G$ with $d = 2$ or $d = 3$ and G a finite abelian group (and also in several other cases), we may apply results or conjectures related to McKay correspondence to say more about such models. In such a case there exists a distinguished basis $\{R_i\}$ of line bundles on X whose sections transform like the characters of G under the action of G [13, 14]. The pairing $\langle\ ,\ \rangle$ of (2.3) leads to a natural basis $\{S_i\}$ of compactly supported K-theory classes obeying $\langle R_i, S_j\rangle = \delta_{ij}$.

Another feature of such models is the existence of a distinguished point in the moduli space called the orbifold point, corresponding to the locus where all exceptional divisors are blown down in the *classical* geometry. It is mirror to the point where all a_i corresponding to the exceptional divisors under the monomial-divisor mirror map vanish. The moduli space is singular at the orbifold point, and in its vicinity it is possible to define a non-simply connected path that leads to 'orbifold monodromy'. For the relation between the S_i and the monodromies the following conjectures exist:

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- At the orbifold point the S_i correspond to the 'fractional branes' [12]; thus they should form a representation of the group of quantum symmetries under orbifold monodromy.
- At some branch of the principal component of the discriminant locus S_0 becomes massless and generates the monodromy around this branch. Often the same is true for the other S_i and different branches [9].

3. D-branes on the resolution of $\mathbb{C}^2/\mathbb{Z}_n$

In this section we want to apply the general strategy outlined above to the case where X is the resolution of $\mathbb{C}^2/\mathbb{Z}_n$. Despite the somewhat unusual nature of the local mirror geometry which is zero dimensional in this case, the machinery of using the GKZ system to calculate quantum volumes and central charges should work.

3.1 The geometry of X and $\tilde{X}$

$\mathbb{C}^2/\mathbb{Z}_n$ can be resolved by the introduction of a set $\{C_1, \ldots, C_{n-1}\}$ of exceptional curves. The classes of these curves generate the Mori cone of the resolution X. Intersection numbers between the C_i are given by $C_i \cdot C_j = -A_{ij}$ where A is the Cartan matrix of $\mathrm{SU}(n)$.

The mirror geometry of this model was mentioned already in [15]. It can easily be determined by toric methods (see [10] for a general discussion of local mirror symmetry and [9] for a derivation in the present context). The polynomial that determines the local mirror is given by

$$P = a_0 y_1^n + a_1 y_1^{n-1} y_2 + \cdots + a_n y_2^n \,. \tag{3.1}$$

The a_i correspond under the monomial-divisor mirror map to the curves C_i where C_0 and C_n can be thought of as the vanishing loci of the coordinates of the orbifolded $\mathbb{C}^2$; therefore we will assume $a_0 \neq 0$ and $a_n \neq 0$. The local mirror $\tilde{X}$ is the vanishing locus of P, i.e. it is just a collection of n points in the $\mathbb{P}^1$ with homogeneous coordinates $(y_1 : y_2)$. It is the analogue of the Seiberg-Witten curve as constructed from local mirror symmetry in [16]. A 'singularity' of $\tilde{X}$ occurs whenever two or more of these points coincide. We will be interested in the complex structure moduli space of $\tilde{X}$ which clearly does not change under redefinitions $y_1 \to \lambda_1 y_1$ and $y_2 \to \lambda_2 y_2$, implying that we should identify the moduli space coordinates $\{a_i\}$ with $\{\lambda_1^{n-i}\lambda_2^i \; a_i\}$ (more precisely, these transformations are the analogues of what we would get for higher dimensional theories where the concept of complex structure makes more sense).

3.2 The GKZ system and its solutions

The GKZ operators corresponding to the Mori cone generators are

$$\partial_{a_0}\partial_{a_2} - \partial_{a_1}^2, \; \partial_{a_1}\partial_{a_3} - \partial_{a_2}^2, \; \ldots \, , \; \partial_{a_{n-2}}\partial_{a_n} - \partial_{a_{n-1}}^2 \,. \tag{3.2}$$

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If we also consider GKZ operators corresponding to elements of the Mori cone that are not generators we find that any equation of the type

$$(\partial_{a_i}\partial_{a_j} - \partial_{a_k}\partial_{a_l})\Pi = 0 \qquad \text{with } i+j = k+l \tag{3.3}$$

belongs to the GKZ system. As far as I am aware this system has been solved only for the case of $n = 2$ [17]. The standard approach towards working with it would be to transform it to the large complex structure coordinates z_i. We find, however, that it is more useful to consider it in terms of the zeroes of P. Defining $b_i = a_i/a_0$ and $x = y_1/y_2$ we get

$$x^n + b_1 x^{n-1} + \cdots + b_n = (x - x_1)(x - x_2)\cdots(x - x_n) = 0\,, \tag{3.4}$$

with

$$b_1 = -x_1 - \cdots - x_n\,,\ b_2 = x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n\,,\ \ \ldots\ ,\ b_n = (-1)^n x_1x_2\cdots x_n\,. \tag{3.5}$$

We have not used the complete freedom of redefinitions, the remaining freedom implying that we should identify $(x_1, \ldots, x_n)$ with $(\lambda x_1, \ldots, \lambda x_n)$ for any $\lambda \neq 0$. In addition we should identify $(x_1, \ldots, x_n)$ with any permuted version of itself. The non-vanishing of a_0 and a_n implies $x_i \neq 0$ and $x_i \neq \infty$, i.e. the moduli space is $(\mathbb{C}^*)^n$ divided by $\mathbb{C}^*$ and S_n.

In order to express the GKZ system in terms of the x_i we first note that the identification of $(x_1, \ldots, x_n)$ with $(\lambda x_1, \ldots, \lambda x_n)$ implies that any solution Π of the GKZ system should be homogeneous in the x_i, i.e. it should obey

$$x_1\frac{\partial\Pi}{\partial x_1} + \cdots + x_n\frac{\partial\Pi}{\partial x_n} = 0\,. \tag{3.6}$$

In the appendix we show that eqs. (3.3) hold if and only if

$$\frac{\partial^2\Pi}{\partial x_i \partial x_j} = 0 \qquad \text{for any } i \neq j\,. \tag{3.7}$$

This set of equations must have a basis of solutions $f_1(x_1), \ldots, f_n(x_n)$ each of which depends only on one of the x_i. By (3.6) it is then clear that $x_i df_i/dx_i$ =const., i.e. $f_i = \alpha + \beta \ln x_i$ with α, β constants. The complete set of solutions of the combined system (3.6, 3.7) is thus generated (redundantly) by

$$\Pi_0 = 1\,, \qquad \Pi_{ij} = \frac{1}{2\pi i}\ln\left(\frac{x_i}{x_j}\right), \quad i \neq j\,. \tag{3.8}$$

3.3 D-brane states and their central charges

We now want to find the precise form of the mirror map by considering the large volume limit of X and the large complex structure limit of $\tilde{X}$. The large complex

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structure coordinates on $\tilde{X}$ are given by $z_i = a_{i-1}a_{i+1}/a_i^2$. Let us assume that there exists a strong hierarchy between the x_i in the sense that $|x_{i+1}/x_i| < \delta$ for all x_i with some small number δ. Then we have $b_i = (-1)^i x_1 x_2 \cdots x_i(1 + \mathcal{O}(\delta))$ and

$$z_i = \frac{a_{i-1}a_{i+1}}{a_i^2} = \frac{b_{i-1}b_{i+1}}{b_i^2} = \frac{x_{i+1}}{x_i}(1 + \mathcal{O}(\delta))\,. \tag{3.9}$$

Thus it is clear that the limit $|x_{i+1}/x_i| \to 0$ for all i is the large volume limit; in this limit

$$\ln z_i = \ln\left(\frac{x_{i+1}}{x_i}\right) + \mathcal{O}(z)\,. \tag{3.10}$$

As the Mori cone is dual to the Kähler cone w.r.t. intersection pairing, a basis $\{[C_i^\vee]\}$ of divisor classes dual to $\{C_i\}$ must form a basis for the Kähler cone. We can now use (2.1) (with $T_i = C_i^\vee$), (2.2) and (3.10) to determine the exact central charges of D-brane states. In terms of a basis $\{p, C_1, \ldots, C_{n-1}\}$ of compact cycles where p denotes the class of a point, we find

$$\begin{aligned} &\mathrm{ch}(S) = p:\ Z^{\mathrm{lv}}(t_i; S) = -1 \quad \Rightarrow \quad Z(S) = -1\,, \\ &\mathrm{ch}(S) = C_i:\ Z^{\mathrm{lv}}(t_i; S) = t_i \quad \Rightarrow \quad Z(S) = \frac{1}{2\pi i}\ln\left(\frac{x_{i+1}}{x_i}\right) + 1\,. \end{aligned} \tag{3.11}$$

If we denote by $\mathcal{O}_p$, $\mathcal{O}_{C_i}$ the structure sheaves of the corresponding cycles as well as the sheaves on X obtained by pushing forward these structure sheaves, we find with the help of the Grothendieck-Riemann-Roch formula

$$\mathrm{ch}(\mathcal{O}_{C_i}) = p + C_i\,, \qquad \mathrm{ch}(\mathcal{O}_p) = p \tag{3.12}$$

and therefore

$$Z(\mathcal{O}_{C_i}) = \frac{1}{2\pi i}\ln\left(\frac{x_{i+1}}{x_i}\right), \qquad Z(\mathcal{O}_p) = -1\,. \tag{3.13}$$

3.4 McKay correspondence and orbifold monodromy

As a brief interlude we will now check that the S_i get permuted cyclically by orbifold monodromy $a_j \to e^{2\pi i\ j/n}a_j$ corresponding to $x_i \to e^{2\pi i/n}x_i$. The line bundles R_i that form the McKay basis of $K(X)$ are given by $R_0 = \mathcal{O}_X$ and $R_i = \mathcal{O}_X(C_i^\vee)$ for $i \geq 1$ [13]. By applying (2.3) we find

$$\mathrm{ch}(S_0) = p + \sum_{i=1}^{n-1} C_i\,, \qquad \mathrm{ch}(S_i) = -C_i \quad \text{for } i > 0 \tag{3.14}$$

leading to

$$Z(S_0) = n - 2 + \sum_{i=1}^{n-1}\frac{1}{2\pi i}\ln\left(\frac{x_{i+1}}{x_i}\right), \qquad Z(S_i) = -\frac{1}{2\pi i}\ln\left(\frac{x_{i+1}}{x_i}\right) - 1 \quad \text{for } i > 0\,. \tag{3.15}$$

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Clearly $\sum_{i=0}^{n} Z(S_i) = -1$ everywhere in the moduli space. The orbifold point is the fixed point of the orbifold monodromy determined by $a_i = 0$ for $1 \leq i \leq n-1$, i.e. the set of x_i is just the set of n-th unit roots (up to a common multiplicative constant). Provided we choose a suitable path from the large complex structure region to the orbifold point, we can get $x_{i+1}/x_i = e^{-2\pi i(n-1)/n}$ and thus $Z(S_i) = -1/n$ for each i. With this choice $Z(S_0) = -(1/2\pi i)\ln(x_1/x_n) - 1$ and orbifold monodromy indeed acts by shifting the S_i cyclically according to the defining representation of $\mathbb{Z}_n$.

3.5 Principal component monodromy

The principal component of the discriminant locus in the moduli space consists of the loci where $x_i = x_j$ for some $i \neq j$. As the moduli space is the space of unordered n-tuples of x_i, i.e. the space of ordered n-tuples divided by the permutation group of the x_i, monodromy around the locus $x_i = x_j$ results in exchanging x_i with x_j. In terms of the solutions (3.8) of the GKZ system, this monodromy acts by

$$\Pi_{ij} \leftrightarrow -\Pi_{ij}\,, \qquad \Pi_{ik} \leftrightarrow \Pi_{jk}\,, \qquad \Pi_{ki} \leftrightarrow \Pi_{kj} \qquad \text{for } k \text{ neither } i \text{ nor } j\,. \tag{3.16}$$

Let us compare this to the root system and Weyl group of SU(n). The roots α_{ij} of SU(n) can be represented as $\alpha_{ij} = e_i - e_j$ where the e_i are linearly independent vectors in an n dimensional space. In particular, the positive roots may be chosen to correspond to $e_{i+1} - e_i$. A Weyl reflection through the plane orthogonal to α_{ij} is determined by $v \to v - \langle \alpha_{ij}, v\rangle_{\mathrm{C}}\, \alpha_{ij}$ with $\langle\ ,\ \rangle_{\mathrm{C}}$ the bilinear pairing that acts on two simple roots by giving the corresponding element of the Cartan matrix. This Weyl reflection is induced by an exchange of e_i and e_j and the complete Weyl group corresponds to the permutation group of the e_i. This shows that the group of principal component monodromies of the GKZ solutions is isomorphic to the Weyl group of SU(n) with the correspondence $\frac{1}{2\pi i}\ln x_i \leftrightarrow e_i$. Note that this fits well with the occurrence of braid group relations [18] in the homological algebra approach to mirror symmetry [19].

This is consistent with the Fourier-Mukai transformation (2.4). We have $Z(\mathcal{O}_{C_i}) = 0$ at $x_i = x_{i+1}$, so we hope that $\mathcal{O}_{C_i}$ is the object generating the monodromy around this locus (according to [18], the $\mathcal{O}_{C_i}$ are spherical and are thus the objects we expect to generate monodromies of this type). Using the pairing (2.3) we find

$$\langle \mathcal{O}_{C_i}, \mathcal{O}_p\rangle = 0 \qquad \text{and} \qquad \langle \mathcal{O}_{C_i}, \mathcal{O}_{C_j}\rangle = -C_i \cdot C_j = A_{ij} \tag{3.17}$$

with A being the Cartan matrix of SU(n). By linearity we also get the right behavior at $x_i = x_j$ with $i > j$ where $Z(\mathcal{O}_{C_{i-1}} + \mathcal{O}_{C_{i-2}} + \cdots + \mathcal{O}_{C_j}) = 0$. Table 1 lists the correspondences between the SU(n) Lie algebra, the GKZ solutions and the D-brane states.

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Lie algebra of SU(n)	GKZ solution / $Z(S)$	D-brane S
Simple root $\alpha_{i+1,i} = e_{i+1} - e_i$	$\Pi_{i+1,i} = \frac{1}{2\pi i}\ln x_{i+1} - \frac{1}{2\pi i}\ln x_i$	$\mathcal{O}_{C_i}$
Positive root $\alpha_{ij} = e_i - e_j$ $= \alpha_{i,i-1} + \cdots + \alpha_{j-1,j}$	$\Pi_{ij} = \frac{1}{2\pi i}\ln x_i - \frac{1}{2\pi i}\ln x_j$ $= \Pi_{i,i-1} + \cdots + \Pi_{j-1,j}$	$\mathcal{O}_{C_{i-1}} + \cdots + \mathcal{O}_{C_j}$
Weyl reflection $e_i \leftrightarrow e_j$ $v \to v - \langle \alpha_{ij}, v\rangle \alpha_{ij}$	$\Pi_{ij} \leftrightarrow -\Pi_{ij}$, $\Pi_{ik} \leftrightarrow \Pi_{jk}$, $\Pi_{ki} \leftrightarrow \Pi_{kj}$	$\mathcal{F} \to \mathcal{F} - \langle S, \mathcal{F}\rangle S$
Weyl group: permutations of e_i	Permutations of x_i	

Table 1: Roots vs. D-branes.

4. Summary and discussion

We have seen that it is possible to describe the moduli space of a resolution of $\mathbb{C}^2/\mathbb{Z}_n$ in terms of an unordered set $\{x_1, \ldots, x_n\}$ of points in $\mathbb{C}^*$, up to a common non-vanishing factor, and to solve the GKZ system explicitly in terms of the x_i. There are two very special points in this moduli space. One of them is the orbifold point where the x_i are just n'th roots of unity. The other one is the point of maximal gauge symmetry enhancement where all of the x_i come together. Our explicit analysis confirms the picture of the orbifold point as the locus in the moduli space where 'fractional branes' with charge $1/n$ times the charge of a D0-brane occur and where McKay correspondence (i.e., connections between the representation theory of the finite orbifolding group and the geometry of the resolution) plays a role. At the point of enhanced gauge symmetry we find that all $n(n-1)/2$ branches of the discriminant locus meet. At each of these branches a specific quantum cycle becomes massless (leading to an SU(2) group), and there is a one to one correspondence of K-theory classes with vanishing central charge at these cycles and combinations of large volume cyles with coefficients that are just the ones that determine the positive roots of SU(n) in terms of the simple ones. This provides a simple answer to the question of which states of wrapping branes should be considered to be independent: there is one state for every (anti-)brane wrapping a cycle in the quantum geometry.

Our picture of the moduli space near the point of gauge enhancement (with all x_i coming together) is very suggestive of a dual type-IIB description where the gauge group would be enhanced by strings becoming massless at the points in moduli space where branes come together. We note, however, that this picture is valid only near the gauge enhancement point and not globally. For example, it could not be used to describe the orbifold point.

The very explicit nature of the present description of the quantum moduli space and the central charges should make it a useful testing ground for many ideas related to finding a precise mathematical formulation of D-brane states. We have seen already how the braid group structures that are expected to arise in the context of derived categories manifest themselves as monodromies. Having a closed formula for the periods should also make it easy to discuss the Π-stability of [20, 21].

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A. Transformation of the GKZ system

In this appendix we show how to transform the GKZ system as formulated in (3.3), involving the a_i, into a system involving the x_i. In order to express ∂_{b_j} in terms of ∂_{x_i} we have to invert the matrix C with $C_{ij} = \partial b_j/\partial x_i$. Differentiating (3.4) by x_i implies

$$\sum_{j=1}^{n} x^{n-j} \frac{\partial b_j}{\partial x_i} = -\prod_{k\neq i}(x - x_k)\,. \tag{A.1}$$

Introducing the matrix D with elements $D_{jk} = x_k^{n-j}$ and the notation

$$\pi_i := \prod_{k\neq i}(x_i - x_k) \tag{A.2}$$

we get $C \cdot D = -\,\mathrm{diag}(\pi_1, \ldots, \pi_n)$ and therefore $(C^{-1})_{ji} = -x_i^{n-j}/\pi_i$ and $\partial_{b_j} = -\sum_i (x_i^{n-j}/\pi_i)\partial_{x_i}$. In terms of the original moduli space coordinates a_i this implies

$$a_0\partial_{a_0} = -\sum_{j=1}^{n} b_j\partial_{b_j} = \sum_{j=1}^{n} b_j \sum_i \frac{x_i^{n-j}}{\pi_i}\partial_{x_i} = \sum_i \frac{\sum_{j=1}^{n} b_j x_i^{n-j}}{\pi_i}\partial_{x_i} = -\sum_i \frac{x_i^n}{\pi_i}\partial_{x_i}\,, \tag{A.3}$$

$$a_0\partial_{a_j} = \partial_{b_j} = -\sum_i \frac{x_i^{n-j}}{\pi_i}\partial_{x_i} \qquad \text{for } j \geq 1\,. \tag{A.4}$$

Now (3.3) can be expressed as

$$\sum_{i=1}^{n} \frac{x_i^p}{\pi_i}\partial_{x_i}\left(\sum_{j=1}^{n} \frac{x_j^q}{\pi_j}\partial_{x_j}\Pi\right) = \sum_{i=1}^{n} \frac{x_i^r}{\pi_i}\partial_{x_i}\left(\sum_{j=1}^{n} \frac{x_j^s}{\pi_j}\partial_{x_j}\Pi\right) \tag{A.5}$$

for all $p, r \in \{0, 1, \ldots, n-1\}$, $q, s \in \{0, 1, \ldots, n\}$ with $p + q = r + s$. Clearly each side of eq. (A.5) can be written as $\sum_j A_j\partial_{x_j}\Pi + \sum_{i,j} B_{ij}\partial_{x_i}\partial_{x_j}\Pi$. We now want to show that the A_j-terms on both sides are the same, i.e. cancel one another. We find

$$A_j = \sum_{i=1}^{n} \frac{x_i^p}{\pi_i}\partial_{x_i}\left(\frac{x_j^q}{\pi_j}\right) = \frac{x_j^q}{\pi_j}\left(\frac{x_j^p}{\pi_j}\left(\frac{q}{x_j} - \sum_{k\neq j}\frac{1}{x_j - x_k}\right) + \sum_{i\neq j}\frac{x_i^p}{\pi_i}\frac{1}{x_j - x_i}\right). \tag{A.6}$$

The last expression can be transformed as

$$\begin{aligned}\sum_{i\neq j}\frac{x_i^p}{\pi_i(x_j - x_i)} &= \sum_{i\neq j}\left(-\frac{x_i^{p-1}}{\pi_i} + \frac{x_j x_i^{p-1}}{\pi_i(x_j - x_i)}\right) = \frac{x_j^{p-1}}{\pi_j} + x_j\sum_{i\neq j}\frac{x_i^{p-1}}{\pi_i(x_j - x_i)} = \cdots \\ &= p\frac{x_j^{p-1}}{\pi_j} + x_j^p\sum_{i\neq j}\frac{1}{\pi_i(x_j - x_i)}\,,\end{aligned} \tag{A.7}$$

where we have made use of $\sum_{i=1}^{n} x_i^k/\pi_i = 0$ for $k < n-1$ (this can be seen by noting that it would have to be a rational function of negative degree but none of

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the possible poles actually is a pole). Thus we arrive at

$$A_j = \frac{x_j^{p+q}}{\pi_j^2}\left(\frac{p+q}{x_j} + \sum_{i\neq j}\frac{\pi_j - \pi_i}{\pi_i(x_j - x_i)}\right), \tag{A.8}$$

which depends on p and q only via $p+q$, ensuring that because of $p+q = r+s$ the corresponding terms drop out from (A.5) which becomes

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{x_i^p x_j^q - x_i^r x_j^s}{\pi_i\pi_j}\frac{\partial^2\Pi}{\partial x_i \partial x_j} = 0\,. \tag{A.9}$$

Choosing $r = p+1$, $q = s+1$ and $s > p$ (equations corresponding to different values can be obtained by linear combinations of these) and reshuffling the summation, we get

$$\begin{aligned} 0 &= \sum_{1\leq i<j\leq n}\frac{x_i^p x_j^{s+1} - x_i^{p+1}x_j^s + x_j^p x_i^{s+1} - x_j^{p+1}x_i^s}{\pi_i\pi_j}\frac{\partial^2\Pi}{\partial x_i\partial x_j} \\ &= \sum_{1\leq i<j\leq n}\frac{(x_i - x_j)^2}{\pi_i\pi_j}(x_i^{s-1}x_j^p + x_i^{s-2}x_j^{p+1} + \cdots + x_i^p x_j^{s-1})\frac{\partial^2\Pi}{\partial x_i\partial x_j} \\ &=: \sum_{1\leq i<j\leq n} M_{ps,ij}\frac{\partial^2\Pi}{\partial x_i\partial x_j} \qquad \text{for all } 0\leq p<s\leq n-1\,. \end{aligned} \tag{A.10}$$

We now want to show that the $\binom{n}{2}\times\binom{n}{2}$ matrix M acting on the $\binom{n}{2}$ vector $\partial^2\Pi/(\partial x_i\partial x_j)$ is regular for generic values of the x_i. This is true iff the matrix $\tilde{M}$ with entries $x_i^{s-1}x_j^p + \cdots + x_i^p x_j^{s-1}$ is regular (the other factor can be pulled out in calculating the determinant of M). The non-vanishing of the determinant of $\tilde{M}$ can be shown inductively. It is true for $n = 2$ and it is not hard to see that

$$\begin{aligned} \det\tilde{M}^{(n)}(x_1 = 0, x_2, \ldots, x_n) &= \det D(x_2,\ldots,x_n)(x_2\cdots x_n)^{n-2}\det\tilde{M}^{(n-1)}(x_2,\ldots,x_n) \\ &\neq 0\,, \end{aligned}$$

where D is the regular matrix we encountered after eq. (A.1). Thus M is regular, and by (A.10) we have completed the proof that (3.3) is equivalent to (3.7).

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